Minimal Model Program Learning Seminar.

Week 6:

- Relative versions,
- How to run the MMP,
- Surface klt singularities.

Theorem (Relative cone theorem): Let $X \xrightarrow{\varphi} Z$ be a projective contraction of alg var over $\mathbb{K}$, $\overline{\mathbb{K}} = \mathbb{K}$ & $\mathrm{char}(\mathbb{K}) = 0$. (X, Δ) klt pair. Then:

excellent $\mathbb{Q}$-scheme.

mixed char dim ≤ 3.

(1) There are countably many $C_j \subseteq X$ s.t $\varphi(C_j) = pt$, $0 < -(K_X + \Delta).C_j < 2\dim X$

and $$\overline{NE}(X/Z) = \overline{NE}(X/Z)_{(K_X+\Delta) \geq 0} + \sum_i \mathbb{R}_{\geq 0}[C_i].$$

(2) For any $\varepsilon > 0$ and φ-ample H,

$$\overline{NE}(X/Z) = \overline{NE}(X/Z)_{(K_X+\Delta+\varepsilon H) \geq 0} + \sum_{i,\ \text{finite}} \mathbb{R}_{\geq 0}[C_i].$$

(3) Let $F \subseteq \overline{NE}(X/Z)$ be a $(K_X+\Delta)$-neg extremal face.

Then there is a unique contraction s.t.

$$\begin{tikzcd} X \arrow[rr, "\mathrm{cont}_F"] \arrow[dr, "\varphi"'] & & Y \arrow[dl, "\varphi_Y"] \\ & Z & \end{tikzcd}$$

$C \subseteq X$ is a mapped to a point iff $[C] \in F$.

(4) $\mathrm{Cont}_F : X \longrightarrow Y$ as in (3). $\mathcal{L}$ is a line bundle on X s.t $\mathcal{L}.C = 0$ for every curve C with $[C] \in F$.

Then there exists $\mathcal{L}_Y$ a line bundle on Y with

$$\mathcal{L} \simeq \mathrm{Cont}_F^* \mathcal{L}_Y.$$

we say that $\mathcal{L}$ descends to Y.

Remark: So far everything is "formal" consequence of Kodaira vanishing & resolution.

Some recent work:

Bhatt & Lurie proved a version of Riemann - Hilbert correspondence in positive char.

Bhatt proved the Cohen-Macaulayness of the integral closure of an excellent Noetherian domain

Using the above the techniques contained above the MMP has been recently generalized in two different directions:

1.- In dimension three in mixed charact (over Spec $\mathbb{Z}$).

Bhatt - Ma - Patakfalvi - Schwede - Tucker - Waldron - Witaszek 2021.

2.- In characteristic zero "most" of the MMP works over an excellent $\mathbb{Q}$ - schem.

Murayama - Lyu 2021

Why MMP relative over base?

MMP to study families of algebraic varieties.

$$\begin{array}{cl} \mathcal{X} & \text{projective smooth} \quad K_X \text{ is ample over } \mathbb{C}^*. \\ \downarrow & \\ \mathbb{C}^* & \end{array}$$

Compactify (arbitrary central fiber, maybe not normal).

$\Downarrow$

Log resolution (many components, K_X not ample over $\mathbb{C}$)

$\Downarrow$

Run MMP over the base.

$$\begin{tikzcd} \mathcal{X} \arrow[r, hook] \arrow[d] & \bar{\mathcal{X}} \arrow[d] \\ \mathbb{C}^* \arrow[r, hook] & \mathbb{C}. \end{tikzcd}$$

so that $K_{\bar{\mathcal{X}}}$ is nef over the base

&

the singularities of $\bar{\mathcal{X}}_0$ are slc.

means.

normalization is lc

&

nodal sing at cod one points.

MMP to study singularities.

$z \in Z$ a log resolution $X \xrightarrow{\varphi} Z$.

$\varphi^*(K_Z) = K_X + \Delta$

Perturb coefficients of Δ:
- If >1, you can decrease to 1
- If <0, you can increase them $\varepsilon > 0$.

Obtain a new boundary B.

Run a MMP for $K_X + B$ over Z, you obtain a partial resolution of singularities which has the singularities of the minimal model program.

Remark: By studying the exceptional divisors of the previous partial resolution & the sing of the MMP, you can understand the singularities of $Z \ni z$.

Flipping contractions & flips:

Definition: $X \xrightarrow{\varphi} W$ is a flipping contraction for (X, Δ) klt if $\mathbb{Q}$-factorial, $\rho(X/W) = 1$, φ is a small birational contr., and $-(K_X + \Delta)$ is ample over W.

(You can have small morphisms with high ρ)

Remark: W is never $\mathbb{Q}$-factorial. K_W is not $\mathbb{Q}$-Cartier.

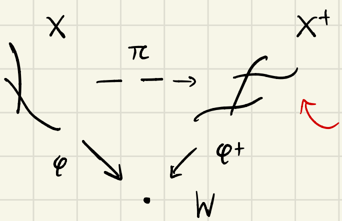
Definition: Let $X \xrightarrow{\varphi} W$ be a flipping contraction for (X, Δ).
We say that $X \overset{\pi}{\dashrightarrow} X^+$ is a flip if it is a small birational map, $K_{X^+} + \Delta^+$ is $\mathbb{Q}$-Cartier ($\Delta^+ = \pi_* \Delta$)
There is a proj morphism $\varphi^+ : X^+ \longrightarrow W$ so that $K_{X^+} + \Delta^+$ is ample over W.

Lemma 1: $f: X \dashrightarrow Y$ small birational map between normal var. $D \in WDiv(X)$. Then

$$H^0(\mathcal{O}_X(D)) \cong H^0(\mathcal{O}_Y(f_* D)).$$

Picture of flip:

K_X-neg curves.



K_{X^+}-positive curves

Lemma 2: Let $X \xrightarrow{\varphi} W$ be a flipping contr. for (X, Δ). Let $X \overset{\pi}{\dashrightarrow} X^+$ be a flip. Then $\rho(X) = \rho(X^+)$ and X^+ is $\mathbb{Q}$-factorial. Moreover, $\rho(X/W) = \rho(X^+/W) = 1$.

Proof: D^+ on X^+, D on X the push-forward.

Find r such that $R \cdot (D + r(K_X + \Delta)) = 0$

Here R is the extremal ray defining the flipping contraction.

We know X is $\mathbb{Q}$-factorial. Hence

$$m(D + r(K_X + \Delta)) \text{ is Cartier for } m \gg 0.$$

$$m(D + r(K_X + \Delta)) \sim \varphi^*(D_W) \text{ for some } D_W \text{ Cartier.}$$

$$mD^+ = m\pi_* D \sim \underbrace{\underbrace{(\varphi^+)^* D_W}_{\text{Cartier}} - \underbrace{(mr)(K_{X^+} + \Delta^+)}_{\text{Cartier}}}_{\text{Cartier.}}$$

$$\begin{tikzcd} X \arrow[rr, dashed, "\pi"] \arrow[dr, "\varphi"'] & & X^+ \arrow[dl, "\varphi^+"] \\ & W & \end{tikzcd}$$

For equality of ρ, we prove that π_* induces an isomorphism between Weil divisors modulo $\sim$. $\square$

Lemma 3: $X \xrightarrow{\varphi} Y$ a projective contraction between normal varieties with $\rho(X/Y)=1$. and $-K_X$ ample over Y. Assume that $\dim(\mathrm{Ex}(\varphi)) = \dim X - 1$. Then φ contracts a unique prime divisor E.

Remark: We call such φ a divisonal contraction.

Proof: Let's say there are two divisors E_1 & E_2.

We can find C_i covering E_i with $C_i \cdot E_i < 0$.

We can find α so that $\underbrace{E_1 + \alpha E_2}_{= E} \equiv_Y 0$.

numerically trivial over φ

Claim: that α is positive.

Assume C_1 does not int E_2

$C_1 \cdot (E_1 + \alpha E_2) = C_1 \cdot E_1 < 0$, pick C_1 general inside E_1

we may assume $E_2 \cdot C_1 \geq 0$. Hence.

$C_1 \cdot E_1 + \alpha E_2 \cdot C_1 = 0$ so $\alpha = \dfrac{-C_1 \cdot E_1}{E_2 \cdot C_1} > 0$.

E is an effective divisor which is contracted so it must be covered by E-negative curves.

We conclude that E_1 must be the only component $\square$.

Proposition: Let $\varphi : X \longrightarrow W$ be a flipping contraction for (X,Δ) klt. The flip exists iff

$$\bigoplus_{m\geq 0} \varphi_* \mathcal{O}_X(m(K_X+\Delta))$$

is a f.g. $\mathcal{O}_W$-algebra. If this is the case, then

$$X^+ := \mathrm{Proj}_W\Big(\bigoplus_{m\geq 0} \varphi_* \mathcal{O}_X(m(K_X+\Delta))\Big).$$

Proof: Assume

$$\begin{tikzcd} X \arrow[rr, dashed, "\pi"] \arrow[dr, "\varphi"'] & & X^+ \arrow[dl, "\varphi^+"] \\ & W & \end{tikzcd}$$

π is small.

$$\bigoplus_{m\geq 0} \varphi_* \mathcal{O}_X(m(K_X+\Delta)) \simeq \bigoplus_{m\geq 0} \varphi^+_* \mathcal{O}_{X^+}(m(K_{X^+}+\Delta^+))$$

by Lemma 1. Moreover $K_{X^+}+\Delta^+$ is ample over W.

Hence,
$$\mathrm{Proj}_W\Big(\bigoplus_{m\geq 0} \varphi^+_* \mathcal{O}_{X^+}(m(K_{X^+}+\Delta^+))\Big) \simeq X^+.$$

Assume $\bigoplus_{m\geq 0} \varphi_* \mathcal{O}_X(m(K_X+\Delta))$ is f.g. $\mathcal{O}_W$-algebra

and define $X^+ = \mathrm{Proj}(\text{———})$.

$X \overset{\pi}{\dashrightarrow} X^+$ is an isom in cod one X.

it could happen that there exists $E \subseteq X^+$ s.t $\pi^{-1}_* E$ is not a div.

$X \xrightarrow{\varphi} W$ is an isomorphism over $X \setminus \mathrm{Ex}(\varphi)$.

$\bigoplus_{m \geq 0} \varphi_* \mathcal{O}_X(m(K_X + \Delta))$ is just sum of copies of the structure sheaf on $X \setminus \mathrm{Ex}(\varphi)$.

Hence $X^+ \overset{\pi^{-1}}{\dashrightarrow} X$ is an isomorphism over $X \setminus \mathrm{Ex}(\varphi)$.

$$\begin{tikzcd} X^+ \arrow[dr, "\varphi^+"'] & \\ & W & X \arrow[l, "\varphi"'] \end{tikzcd} \qquad E \subset X^+$$

E is a mapped to a higher codim cycle by φ^+.

$$\varphi^+_* \mathcal{O}_{X^+}(1) \simeq \varphi_* \mathcal{O}_X(m(K_X + \Delta)) \simeq \mathcal{O}_W(m(K_W + \varphi_* \Delta))$$

for some $m > 0$. Since E is exc over W, we have

$$\mathcal{O}_W(tm(K_W + \varphi_* \Delta)) = \varphi^+_* \mathcal{O}_{X^+}(t) \underset{=}{\subsetneq} \varphi^+_* \mathcal{O}_{X^+}(t)(E).$$

We have a natural inclusion

$$\varphi^+_* \mathcal{O}_{X^+}(t)(E) \hookrightarrow \mathcal{O}_W(tm(K_W + \varphi_* \Delta))$$

No contracted divisors by φ^+. Thus, π is small.

By Lemma 2, $\rho(X/W) = \rho(X^+/W) = 1$. $\square$

Finite generation of the canonical ring:

Conj: Let $X \xrightarrow{\varphi} Z$ proj morphism (X, Δ) klt.
Then $\bigoplus_{m \geq 0} \varphi_* \mathcal{O}_X(m(K_X + \Delta))$ is a f.g $\mathcal{O}_Z$-algebra.

Rmk: X smooth proj variety. $\bigoplus_{m \geq 0} H^0(X, \mathcal{O}_X(mK_X))$ is finitely generated over $\mathbb{K}$. (is a part case of conj).

How to run the MMP:

X_i to be $\mathbb{Q}$-factorial.

1.- (X_i, Δ_i) klt pair, $X_i \longrightarrow Z$ proj morph.
If $K_{X_i} + \Delta_i$ nef over Z, then we stop and call this a minimal model over Z.

If $K_{X_i} + \Delta_i$ is not nef over Z, we consider an extremal ray R in $\overline{NE}(X_i / Z)$ which is $(K_{X_i} + \Delta_i)$-neg.

2.- Let $X_i \longrightarrow W$ be the contraction defined by R.
a) $\dim(W) < \dim(X_i)$, $-K_{X_i}$ ample over W and the general fiber klt. Hence, the general fiber is klt Fano.
In this case we stop and call this a Mori fiber space.

b) $\dim X = \dim W_i$ and $X \xrightarrow{f} W$ contains a divisor in its exc locus. By Lemma 3 this is a divisorial contraction W is $\mathbb{Q}$-factorial, $\rho(W/Z) = \rho(X/Z) - 1$.

We denote $X_{i+1} := W$ & $\Delta_{i+1} := f_*(\Delta_i)$.

Return to step 1.

Remark: Using neg Lemma, we can prove (X_{i+1}, Δ_{i+1}) is klt.

c) $\dim(X) = \dim(W)$ and $X \to W$ small bir map.

"We find the flip" $X \overset{\pi}{\dashrightarrow} X^+$ and define

$$X_{i+1} = X^+ \text{ and } \Delta_{i+1} = \pi_* \Delta_i.$$

By Lemma 2 & neg lemma, X_{i+1} is $\mathbb{Q}$-fact provided that X_i is $\mathbb{Q}$-factorial and $\rho(X_i/Z) = \rho(X_{i+1}/Z)$.

Return to step 1.

Possible outcomes: Minimal Model or Mori fiber space (MFS).

$$\text{Minimal Model} \xrightarrow{\text{Abundance}} \text{Canonical Model}$$

Singularities when running the MMP:

Proposition: Let (X,Δ) be a log canonical pair (resp. klt, canonical, terminal). Let $(X,\Delta) \overset{\pi}{\dashrightarrow} (X',\Delta')$ be a step of the $(K_X+\Delta)$-MMP. Then (X',Δ') is log canonical (resp. klt, canonical, terminal).

Let E be a prime divisor over X whose center is contained in $\mathrm{Ex}(\pi)$. Then, we have an inequality

$$a_E(X',\Delta') > a_E(X,\Delta).$$

$$\begin{tikzcd} X \arrow[rr, dashed, "\pi"] \arrow[dr, "e"'] & & X^+ \arrow[dl, "\varphi^+"] \\ & W & \end{tikzcd}$$

flip

Proof: Let $p: Y \longrightarrow X$ be a log resolution of (X,Δ) which dominates X'. Let $q: Y \longrightarrow X'$ be the corresponding projective birational morphism.

Write $p^*(K_X+\Delta) = q^*(K_{X'}+\Delta') + F_1 - F_2$,

where F_1 & F_2 are effective with disjoint support.

The divisor $F_1 - F_2$ is q-exceptional,

by the projection formula it is anti-nef over X'.

Since the push-forward of $F_1 - F_2$ to X' is eff, we conclude that $F_2 = 0$, so the first statement holds. Indeed, for any $E \subseteq Y$ prime we have:

$$\begin{aligned} a_E(X', \Delta') &= a_E(X, \Delta) + \mathrm{coeff}_E(F_1) \\ &\geq a_E(X, \Delta). \end{aligned} \tag{1}$$

Now, we want to prove that if $c_X(E) \subseteq \mathrm{Ex}(\pi)$, then (1) is strict. Equivalently that $E \subseteq \mathrm{supp}(F_1)$.

Note that $c_{X'}(E) \subseteq \mathrm{Ex}(\pi^{-1})$. Applying the 2nd part of negativity Lemma we get that either

i) $E \subseteq \mathrm{Supp}(F_1)$, or

ii) $E \cap \mathrm{Supp}(F_1) = \emptyset$.

Take $C \subseteq Y$ and mapping to a point in X' so that $E.C < 0$. Hence, we conclude that

$$p^*(K_X + \Delta).C > 0$$

This leads to a contradiction because $-p^*(K_X + \Delta)$ is nef over W.

$\square$.

Surface singularities of the MMP:

Theorem: The following statements hold:

1. $(x \in X)$ is a surface klt singularity $\Longleftrightarrow$ $(x \in X)$ is the quotient of $(0 \in \mathbb{C}^2)$ by a finite subgroup of $GL_2(\mathbb{C})$.

2. $(x \in X)$ a canonical surface singularity $\Longleftrightarrow$ $(x \in X)$ is the quotient of $(0 \in \mathbb{C}^2)$ by a finite subgroup of $SL_2(\mathbb{C})$.

3. $(x \in X)$ is terminal surface sing $\Longleftrightarrow$ x is a smooth point X

Idea: K_X is $\mathbb{Q}$-Cartier, we can take its index one cover.

$$G \curvearrowright Y \xrightarrow{\pi} X \qquad X = Y/G.$$

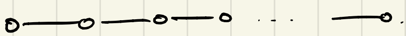
finite Galois quasi-étale

K_Y is a Cartier divisor, Y is again klt and since K_Y is cartier its log discrepancies are in $\mathbb{Z}_{>0}$ so is canonical

Du Val singularities:

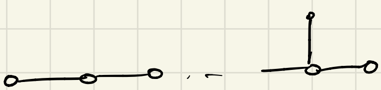
Theorem: Let $x \in X$ be a canonical surface sing. Then $x \in X$ has embedding dimension three. Moreover, up to analytic change of coordinates, the following is a complete list of the possible singularities:

A: A_n $(n \geq 0)$ has eq $x^2 + y^2 + z^{n+1} = 0$ and dual graph



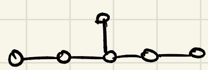
n vertices

D: D_n $(n \geq 4)$ has eq $x^2 + y^2 z + z^{n-1} = 0$ and dual graph

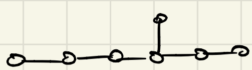


n vertices.

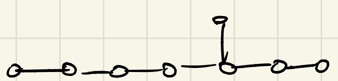
E: E_6: $x^2 + y^3 + z^4 = 0$



E_7: $x^3 + y^3 + yz^3 = 0$



E_8: $x^2 + y^3 + z^5 = 0$



Idea of the proof: Study dual graph of the resolution & use W. preparation theorem to write down the eqs.